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A METHOD OF CALCULATING FLUTTER OF A CONICAL NOZZLE WITH INTERNAL SUPERSONIC FLOW

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INTRODUCTION

The problem of vibrations of a conical elastic shell, treated in many important works, has not been so far, to the author's knowledge, solved satisfactorily, because of its great mathematical complexity. The flutter of a divergent nozzle considered as a conical shell is still more involved due to the additional difficulties connected with the necessary determination of the nonstationary internal flow.

The vibrations of a conical shell in vacuum were analyzed by Strutt,[9] using energy methods, for the case of a conical shell, built in at its smaller base and free at the other end, assuming that the mean surface is inextensible. Breslavskii, using energy method and replacing the conical shell by an equivalent cylindrical shell of appropriate diameter, has investigated the influence of extension of the conical shells mean surface on its natural frequencies. Grigoluk,[4] using also energy methods, has determined the natural frequencies of a conical shell simply supported at the periphery of both ends.

In some papers of particular interest to acoustics[7,8] concerning vibrations of conical loudspeaker membranes or shells, attempts were made to determine their natural frequencies making different simplifications in the shell equations and boundary conditions, using either energy methods or trying solutions in form of infinite series. Shulman[10] in his Sc.D. thesis has used energy and Galerkin methods to analyze the vibrations of conical shells in vaccum. Using Galerkin's method Shulman has also solved the flutter problem of a conical shell in external flow using the piston theory approximation for the aerodynamic pressure.

Dzygadlo[2] has given a solution of the flutter problem of a pointed conical shell in an external supersonic flow, deducing aerodynamic forces from the perturbation potential equations and solving the shell equations by the method of finite Fourier transformations.

Niesytto and Sep[4] investigated flutter of a cylindrical shell with internal flow. The flow is found from a perturbation potential, whose solution, assuming constant density, is given in the form of an infinite series with coefficients determined from the solution of an infinite set of equations obtained from the requirement of satisfying the aerodynamic boundary conditions. The solution of the shell equation is also obtained in the form of an infinite series with coefficients that are solutions of an infinite set of equations obtained from the boundary conditions of the shell. The critical parameters can be calculated limiting the characteristic determinant to a finite number of terms. This method seems rather complicated and as the pressure is expressed in the form of

$$\rho = \frac{K(r,\omega,U)}{\dfrac{\partial K(r,\omega,U)}{\partial r}} f(w,x,t)$$

where K is Bessel's function depending on the flow velocity and vibration frequency, the possibility of encountering in practice the case of the flow parameters being such, that the denominator becomes zero, requires further clarification.

FLOW THROUGH A DIVERGENT NOZZLE WITH OSCILLATING WALLS

In order to determine the critical flutter parameters it is necessary to find the nonstationary flow through a divergent nozzle with harmonically oscillating walls. This problem was solved by the author[3] and the method will be briefly summarized here.

Assuming that the flow can be described by a perturbation potential and that the nozzle section changes sufficiently smoothly into the divergent shape so that there are no velocity singularities the disturbance potential φ equation takes the following form:

$$(M^2 - 1)\frac{\partial^2 \varphi_n}{\partial x^2} - \frac{\partial^2 \varphi_n}{\partial R^2} - \frac{1}{R}\frac{\partial \varphi_n}{\partial R} + 2i\omega M^2 \frac{\partial \varphi_n}{\partial x} + \left(\frac{n^2}{R^2} - M^2\omega^2\right)\varphi_n = 0 \quad (1)$$

Calling $w(x,\alpha,t) = \sum_n w_n(x) \cos n\alpha e^{i\omega t}$ the normal wall displacement, the boundary conditions at the nozzle walls, whose static shape is given by $R_{ws} = 1 + f(x)$, are:

$$\left[\frac{\partial \varphi_n(x,R)}{\partial R} - \frac{\partial f(x)}{\partial x}\frac{\partial \varphi_n(x,R)}{\partial x}\right]_{R=R_{ws}} = i\omega\omega_n(x) + \left[1 + \frac{\partial \phi_s(x,R)}{\partial x}\right]\frac{d\omega_n(x)}{dx} \quad (2)$$

and on the characteristics $0_1 1$, $0_2 1$ at the entry to the divergent nozzle, i.e., on

$$\pm R = 1 - x\tan\beta \quad \text{for} \quad 0 \leqslant x \leqslant \cot\beta \quad (3)$$

$$\varphi_n = 0$$

Having solved the potential Eq. (1) for the above boundary condition the oscillating pressure amplitude can be found, for each value of the cross-section vibration mode n, from the well known linearized relation:

$$p = -\rho_0 U_0^2 \left(\frac{\partial \varphi_n}{\partial t} + \frac{\partial \varphi_n}{\partial x}\right) \tag{4}$$

To find the flow potential we will transfer to the plane of characteristics (ξ, η) using the known transformation

$$\begin{aligned} x &= \xi \cos \beta + \eta \sin \beta \\ R &= 1 - \xi \sin \beta + \eta \cos \beta \end{aligned} \tag{5}$$

where $\tan \beta = -\dfrac{1}{\sqrt{M^2 - 1}}$

The potential equation and the boundary conditions will be transformed accordingly into ξ, η variables and the nozzle walls will be given by relations η_{ω}, (ξ) and $\eta_{\omega_2}(\xi)$.

Dividing the whole nozzle area into regions such as 0_1 1 2, 1 2 4 3, 3 4 6, . . . limited by the characteristics, it can be noted that for the regions having the nozzle wall as one of its boundaries, e.g., $0_2 13$, we have a modified Picard problem and for the ones limited by characteristic lines only, e.g., 1243, we have a Darboux problem.

The solution of Eq. (1) taking into account the boundary conditions [Eqs. (2) and (3)] transformed into the characteristics plane can hence be solved by reducing to Volterra's integral equations whose Riemann functions can be found using known methods.

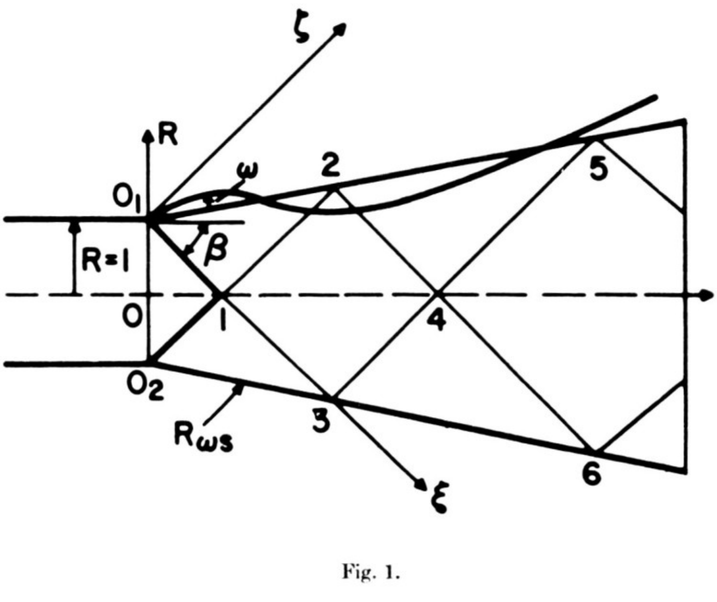

Fig. 1.

DETERMINATION OF CRITICAL FLUTTER VELOCITY

As can be seen from the previous section, the determination of the pressure on the nozzle wall can be made only after assuming the wall displacement mode. This limits the possible methods of solution of the elastic shell equations taking into account the aerodynamic loads to the energy and Galerkin methods.

The equation of small shell vibrations in ξ, ζ coordinates, taking into account only the normal components of the inertia forces, have the following form:

$$\nabla^4 F - Eh\nabla_k^2 w = 0$$

$$D\nabla^4 w + \nabla_k^2 F + \rho_{sh} h \frac{\partial^2 w}{\partial t^2} - p = 0 \qquad (6)$$

where F = stress function

w = normal component of shell displacement

$$\nabla^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\xi}\left(\frac{B}{A}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{A}{B}\frac{\partial}{\partial\zeta}\right)\right]$$

$$\nabla_k^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\xi}\left(\frac{B}{A}k_2\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{A}{B}k_1\frac{\partial}{\partial\zeta}\right)\right] \qquad (7)$$

A, B = first fundamental magnitudes of the shell midsurface

$k_1 = \frac{1}{R_1}$, $k_2 = \frac{1}{R_2}$ = shell-midsurface curvatures

$D = \frac{Eh^3}{12(1 - \nu^2)}$ – bending rigidity

p = aerodynamic load

The stress, F, and displacement, w, functions must satisfy corresponding boundary conditions, which using the "technical" shell theory and Oniashvili's[6] approximations are:

Built-in, or simply supported, edges;

normal displacement:

$$w = 0 \qquad (8)$$

longitudinal displacement:

$$u = 0 \qquad (9)$$

which according to Ref. 6 is equivalent to

$$\frac{\partial^3 F}{\partial \xi^3} = 0$$

circumferential displacement:

$$v = 0 \tag{10}$$

which according to Ref. 6 is equivalent to

$$\frac{\partial^2 F}{\partial \zeta^2} = 0$$

and for the case of the built-in edge:

$$\frac{\partial w}{\partial \xi} = 0 \tag{11}$$

or in the case of the simply supported edge:

$$M_1 = D\left(\frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \zeta^2}\right) = 0 \tag{12}$$

Free edges: The generalized, according to Kirchhoff, shearing force:

$$Q_1{}^* = -D\left[\frac{\partial^3 w}{\partial \xi^2} + (2 - \nu)\frac{\partial^3 w}{\partial \xi \partial \zeta^2}\right] = 0 \tag{13}$$

Bending moment:

$$M_1 = D\left(\frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \zeta^2}\right) = 0 \tag{14}$$

longitudinal force:

$$T_1 = \frac{\partial^2 F}{\partial \xi^2} = 0 \tag{15}$$

circumferential force:

$$S = -\frac{\partial^2 F}{\partial \xi \partial \zeta} = 0$$

Generally speaking, two sets of functions, F_{mn} and w_{mn}, satisfying the chosen boundary conditions, can be assumed and subsequently using the variational energy method the characteristic determinant, from which the critical parameters are deduced, can be obtained. Due to the tremendous amount of work involved in calculating the energy and generalized forces this method seems impractical.

The orthogonalization variational method used by Oniashvili[6] to analyze shell vibrations consisting in the determination of the stress F and displacement w functions using two orthogonal function sets satisfying the required boundary conditions and at the same time mutually orthogonal, cannot be used in the present case, as such sets in the case of conical shells even with partially fulfilled boundary conditions are not mutually orthogonal.

From the above considerations it appears that the only practical method is Galerkin's based on a set of functions satisfying chosen but not necessarily all boundary conditions.

From the set of Eq. (6) following, i.e., using the identity deduced by Shulman[10]:

$$[r^4\nabla^4(\cdots)]_{rr} = r^2\nabla^4[r^2(\cdots)_{rr}] \tag{16}$$

in the case of a conical shell using spherical coordinates for which

$$\begin{aligned} &\xi = r, \quad A = 1, \quad R_1 = \infty, \quad k_1 = 0 \\ &\zeta = \alpha, \quad B = r \sin \Theta \\ &R_2 = \frac{r}{\cot \Theta}\, k_2, \quad = \frac{\cot \Theta}{r} \end{aligned} \tag{17}$$

the following single equation is obtained:

$$Dr^2\nabla^4(r^3\nabla^4 w) + Eh \cot^2\Theta\, [r^3 w_{rr}]_{rr} + \zeta_{sh} h r^2 \nabla^4 (r^3 w'') - r^2\nabla^4 r^4 p = 0 \tag{17a}$$

where

$$\nabla^2(\cdots) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial(\cdots)}{\partial r}\right] + \frac{1}{r^2 \sin^2\Theta}\frac{\partial^2(\cdots)}{\partial \alpha^2} \tag{18}$$

We assume then, that the displacement can be expressed by the series:

$$w = \sum_{mn}\left[C_{mn} \sin \frac{m\pi(r - r_0)}{r_1 - r_0} + f_n \frac{r - r_0}{r_1 - r_0}\right] \cos n\alpha e^{i\omega t} \tag{19}$$

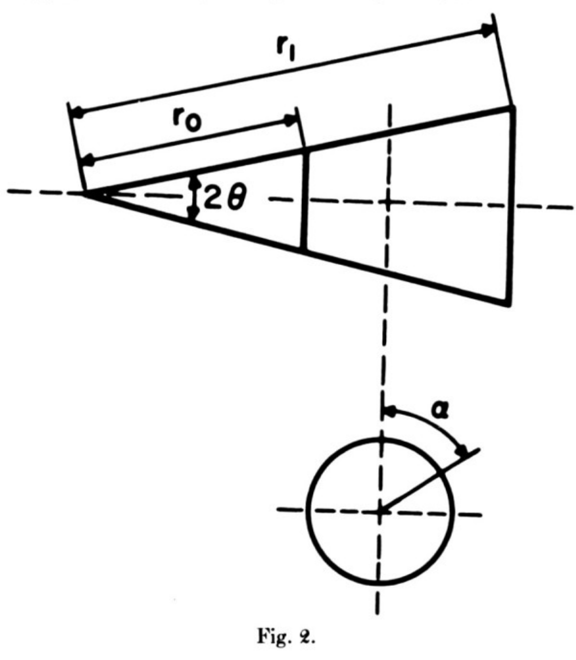

Fig. 2.

where fn is determined from the condition, that the generalized Kirchhoff force at the free end is zero, i.e.,

$$\left[\frac{\partial^3 w}{\partial r^3} + (2 - \nu) \frac{\partial^3 w}{\partial r \partial \alpha^2} \right]_{r=r_1} = 0 \tag{20}$$

wherefrom after transformations:

$$f_n = \frac{r_1 - r_0}{(2 - \gamma)n^2} \sum_m C_{mn} \left[\left(\frac{m\pi}{r_1 - r_0} \right)^3 + (2 - \nu) \left(\frac{m\pi}{r_1 - r_0} \right) n^2 \right] \tag{21}$$

and

$$w = \sum_{mn} C_{mn} w_{mn} e^{i\omega t} = \sum_{mn} C_{mn} \left\{ \sin \frac{m\pi(r - r_0)}{r_1 - r_0} \right.$$
$$\left. - \frac{(r - r_0)}{(2 - \nu)n^2} \left[\left(\frac{m\pi}{r_1 - r_0} \right)^3 + (2 + \nu) \left(\frac{m\pi}{r_1 - r_0} \right) n^2 \right] \right\} \cos n\alpha e^{i\omega t} \tag{22}$$

The assumed displacement satisfies exactly only two boundary conditions, zero displacement at the fixed end and zero shearing force at the free end. According to Oniashvili[6] in the case of thin shells Poisson's coefficients can be taken equal to zero, $\nu = 0$ in the moment boundary conditions. In this case Eq. (22) will satisfy additionally two boundary "moment" conditions which physically means that one edge is hinged, the other is free.

In Eq. (17a) there is the unknown aerodynamic pressure, which is determined for each w_{kl} and assumed velocity and vibration frequency using the method previously summarized. The pressure distribution, p_{kl}, calcuated in this way is then developed in a Fourier series in

$$\sin \frac{m\pi(r - r_0)}{r_1 - r_0} \quad \text{and} \quad \cos \frac{m\pi(r - r_0)}{r_1 - r_0} \quad \text{i.e.,}$$

$$P_{kl} = \sum_{mn} s_{klmn} \sin \frac{m\pi(r - r_0)}{r_1 - r_0} \cos n\alpha + \sum c_{klmn} \cos \frac{m\pi(r - r_0)}{r_1 - r_0} \cdot \cos n\alpha \tag{23}$$

For the assumed vibration mode [Eq. (22)] and the chosen number of values of m and n the aerodynamic loads will be expressed by the following formula:

$$p = \left[\sum_{mn} \sin \frac{m\pi(r - r_0)}{r_1 - r_0} \cos n\alpha \sum_{kl} s_{klmn} \right.$$
$$\left. + \sum_{mn} \cos \frac{m\pi(r - r_0)}{r_1 - r_0} \cos n\alpha \sum_{kl} c_{klmn} \right] e^{i\omega t}, \; m = 1,2,3 \cdots \; n = 0,1,2 \cdots \tag{24}$$

where the Fourier coefficients s_{klmn} and c_{klmn} are functions of frequency and velocity.

The procedure of determination of critical parameters is hence as follows.

The quantity of terms, $m + n$ which will be used in the calculation is chosen and for this number of terms $k + l = m + n$ the aerodynamic load is determined for a given pair of values of frequency and velocity. Then we find the coefficients of the corresponding Fourier series s_{klmn} and c_{klmn}.

Substituting Eqs. (22) and (24) in (17a) multiplying Eq. (17a) by respective $w_{m,n}$ and integrating over the conical shell surface a set of $m + n$ algebraic equations for the c_{mn} unknown coefficients is obtained. The critical parameters will be determined from the condition that the characteristic determinant must be equal to zero if the chosen set of frequency and velocity magnitudes correspond to the critical condition. The process of finding the zero value of the characteristic determinant is quite laborious, as the effect of changing the velocity and frequency on the value of the characteristic determinant must be found by a trial and error method. Unfortunately, a simpler method of finding the critical values is not known.

The calculation may be greatly simplified if we limit ourselves to a single value of n and a certain number of longitudinal waves m as Shulman[10] has done for the case of external flow.

It is regretted that no results of actual numerical calculations are available to be presented.

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Discussor: David John Johns, College of Aeronautics

My impression on reading and hearing this paper is that the lecturer has taken considerable effort to make the calculation of the aerodynamic forces exact, but I feel that the structural terms are not as accurate. A more general set of shell deformation equations are probably necessary to get worthwhile results, equations which consider the in-plane inertia forces as well as the radial inertia forces. Such forces have been shown to be important in studies for circular cylindrical shells. Would the lecturer care to comment?

If, however, the simpler form of equations are to be solved [i.e., Eq. (6)], then instead of the method of analysis presented, has the lecturer considered the possibility of assuming first the form of the function w and using the first of equations in Eq. (6) to derive the form of the function f? These two functions can then be substituted into the second of equations in (6) which can be solved by Galerkin's method.

Author's reply to discussion:

The method given in the paper is an attempt—the first known to the author—to solve the difficult problem presented. It was considered sufficient, as a first approximation, to use simpler shell-deformation equations and later on to the neglected terms. Unfortunately, there is no experience available yet on the effect of the neglected inertia terms and I agree that this must be investigated.

As to the proposed approach to the solution of the set of equations in Eq. (6), it is a valuable and interesting suggestion.

It may be worth noting that the aerodynamic forces are not calculated exactly, but also by an approximate method which is quite complicated.